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STABILITY OF PERIODIC OSCILLATIONS IN SYSTEMS WITH MILD AND STRICT NONLINEARITY*

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Nonautonomous weakly dissipative systems with one degree of freedom and mild and strict nonlinearity are considered without constraints on the degree of nonlinearity and magnitude of perturbing effects. A new method is proposed for the investigation of stability of periodic oscillations. It is based on estimates of the spacing of zeros of solutions of the variational equation. Conditions of stability and instability of fundamental periodic oscillations, determined by the form of perturbing effects and the character of nonlinearity, are estiablished. Theorems on the number of investigated periodic solutions and their amplitude-frequency characteristics are proved.

The majority of investigations of stability of periodic osillations was carried out by asymptotic methods or the method of small parameter, and related to quasilinear and quasi-Liapunov systems (see /1—3/). Some sufficient conditions of asymptotic stability as a whole were obtained using the Liapunov functions with very strict constraints on the degree of nonlinearity /4/. These and other known data cannot be used when the nonlinear and non-autonomous terms are fairly large. Some systems of this type were investigated by approximate analytic methods and by simulation /5,6/. Rigorous solutions are obtained in the present paper for a wide class of systems, without constraints on the magnitude of nonlinear and nonautonomous terms.

1. The nonautonomous nonlinear system

$$x^{\cdot\cdot} + \mu\varphi(x, x^{\cdot}, \omega t) + f(x) = p(\omega t), \quad \mu > 0, \ f(x) = -f(-x), \ f(x)x \geqslant 0 \text{ for } |x| \leqslant c < \infty \tag{1.1}$$

$$\varphi(x, x^{\cdot}, \tau) = \varphi(x, x^{\cdot}, \tau + 2\pi), \quad p(\omega t) = \sum_k p_k \cos k\omega t, \ k = 1, 3, 5, \ldots$$

with one degree of freedom is considered on the assumption that $f(x)$ and $\varphi(x, x^{\cdot}, \omega t)$ are differentiable with respect to their arguments and that

$$\int_0^T \varphi_{x^{\cdot}}(x(t), x^{\cdot}(t), \omega t)\, dt > 0, \quad T = \frac{2\pi}{\omega} \tag{1.2}$$

where $x(t)$ is any function of the form

$$x(t) = \sum_k x_k \cos k\omega t, \quad k = 1, 3, 5, \ldots \tag{1.3}$$

The term $\mu\varphi(x, x^{\cdot}, \omega t)$ defines dissipative forces whose smallness is taken into consideration by parameter μ. Note that in dissipative systems, as a rule, $\varphi_{x^{\cdot}}(x, x^{\cdot}, \omega t) > 0$, i.e. condition (1.2) is satisfied.

It is assumed that when $x > 0$, $f(x)$ is convex (strict nonlinearity) or concave (mild nonlinearity). In the latter case $f(x)$ may change its sign at some $x = c$.

Let $x(t)$ be a periodic solution of Eq. (1.1) when $\mu = 0$.

$$x^{\cdot\cdot} + f(x) = p(\omega t) \tag{1.4}$$

We consider the respective variational equation

$$y^{\cdot\cdot} + a(t)y = 0, \quad a(t) = f_x(x(t)) \tag{1.5}$$

If its multipliers are not unity, there exists for fairly small μ a unique periodic solution $x(t, \mu)$ of Eq. (1.1) which reduces to $x(t)$ when $\mu = 0$ /3/. The respective variational equation is of the form

$$y^{\cdot\cdot} + b(t, \mu)y^{\cdot} + a(t, \mu)y = 0 \tag{1.6}$$

$$b(t, \mu) = \mu\varphi_{x^{\cdot}}(x(t, \mu), x^{\cdot}(t, \mu), \omega t), \ a(t, \mu) = f_x(x(t, \mu)) + \mu\varphi_x(x(t, \mu), x^{\cdot}(t, \mu), \omega t)$$

For small μ we have $x(t, \mu) \approx x(t)$, $a(t, \mu) \approx a(t)$, and $b(t, \mu) \approx 0$. If the multipliers of Eq. (1.5) are real and not 1 or -1, the solution $x(t)$ is unstable. The corresponding solution $x(t, \mu)$

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*Prikl.Matem.Mekhan.,44,No.4,640-649,1980 is obviously also unstable when μ is small. If the multipliers are complex, then with condition (1.2) satisfied and a small μ the multipliers of Eq. (1.6) lie within the unit circle /3/. Because of this, the stability of the trivial solution of Eq. (1.5) ensures under these conditions an asymptotic stability of solution $x(t, \mu)$ (but not that of $x(t)$).

Let us assume that the momentum of perturbing force is nonnegative in the first quarter of the period, i.e.

$$P(t)=\int_0^t p(\omega s)\,ds \geqslant 0 \quad \text{for } 0 \leqslant t \leqslant \frac{\pi}{2\omega} \tag{1.7}$$

This condition is satisfied if $p(\omega t)$ changes its sign twice in a period, since then

$$p(\omega t) \geqslant 0 \quad \text{for } 0 \leqslant t \leqslant \pi/2\omega \tag{1.8}$$

Let $x(t)$ be a solution of Eq. (1.4) with initial conditions $x(0) = -A > -c\ (A > 0)$, $x^{\cdot}(0) = 0$. By virtue of (1.1) and (1.7), for large values of parameter

$$x(t) < 0, \quad x^{\cdot}(t) = \int_0^t [p(\omega s) - f(x(s))]\,ds > 0 \quad \text{for } 0 < t \leqslant \frac{\pi}{2\omega}$$

Hence, as ω decreases it is possible to find an $\omega(A)$ such that $x(\pi/(2\omega)) = 0$. The corresponding solution $x(t)$ is obviously periodic of period $T = 2\pi/\omega$, and

$$x(t) = -x(\pi/\omega - t), \quad x(t) = x(-t), \quad x^{\cdot}(t) > 0 \quad \text{for } 0 < t \leqslant \pi/2\omega \tag{1.9}$$

which implies that $x(t)$ is of form (1.1) and monotonically varies between the extremal values $-A$ and A.

Thus, when condition (1.7) is satisfied, there exists a solution of form (1.9) for any $A \in (0, c)$. It can be shown that $\omega(A) \to \infty$ as $A \to 0$. If $f(x)$ is an increasing function, then $\omega(A) \to \omega_0(A)$ as $A \to \infty$, where $\omega_0(A)$ is the amplitude-frequency characteristic of free oscillations (skeleton curve).

Below, we consider besides (1.3) and (1.9) periodic solutions of the type

$$x(t) = -x(\pi/\omega - t), \; x(t) = x(-t), \; x(t) \geqslant 0 \text{ for } 0 \leqslant t \leqslant \pi/2\omega \tag{1.10}$$

Solution (1.10) is in "the opposite phase" to (1.9) and changes its sign twice in a period, but is generally nonmonotonic on $[0, \pi/\omega]$. If $f(x)$ increases and $f(x) > p_* = \max p(t)$ for $x > x_*$, such solution certainly exists for any $A > A_*$, where A_* is the root of equation

$$F(A) - p_* A = 0, \quad F(A) = \int_0^A f(x)\,dx$$

Indeed, it is possible to show that when $A > A_*$,the phase trajectory $v = x^{\cdot} = 0$ of Eq. (1.4) issuing from point $x = A$ at $p = p_*$ (and, consequently, for any $p(t) \leqslant p_*$) satisfies the inequality $v(x) < 0$ on $[0, A)$. We thus have $x^{\cdot}(t) < 0$ for any ω and $x(t) > 0$. Hence there exists an $\omega(A)$ such that $x(\pi/(2\omega)) = 0$. The respective solution is of form (1.10) and $x(t)$ monotonically decreases on $[0, \pi/\omega]$.

2. We first consider a system with mild nonlinearity, and assume that $f(x)$ is a nondecreasing function, i.e.

$$f_x(x) \geqslant 0 \tag{2.1}$$

Let $A(\omega)$ be the amplitude-frequency characteristic of the solution of form (1.9), $\omega_* = \inf \omega(A)$ when $A \in (0, \infty)$, and $A_0(\omega)$ be a skeleton curve.

Theorem 1. The system (1.4), (1.7), (2.1) with mild nonlinearity has for any $\omega \in (\omega_*, \infty)$ a unique periodic solution of form (1.9). Function $A(\omega)$ monotonically decreases and satisfies the inequality $A(\omega) > A_0(\omega)$. The corresponding solution $x(t, \mu)$ of Eq. (1.1) with condition (1.2) and a fairly small μ is asymptotically stable.

Proof. Let us first prove that the trivial solution of Eq. (1.5), where $x(t)$ is of form (1.9), is stable.

In a system with mild nonlinearity $f_x(x)$ is a nonincreasing function which for $x > 0$ is even. Hence it follows from (1.9) that $a(t)$ is periodic of period $\theta = \pi/\omega$, increases (does not decrease) on $[0, \theta]$, and satisfies the relation $a(t) = a(\theta - t)$.

We multiply (1.4) by $y^{\cdot}(t)$ and integrate from 0 to t. After some simple transformations with allowance for (1.5), we obtain

$$x^{\cdot}(s)\,y^{\cdot}(s)\big|_0^t = \int_0^t p(\omega s)\,y^{\cdot}(s)\,ds - f(x(s))y(s)\big|_0^t \tag{2.2}$$

or

$$x^{\cdot}(s)\,y^{\cdot}(s)\big|_0^t = \int_0^t P(s)\,a(s)\,y(s)\,ds + P(s)\,y(s)\big|_0^t - f(x(s))\,y(s)\big|_0^t \tag{2.3}$$

We denote by $y_1(t)$ the solution of (1.5) that satisfies conditions $y_1(0) = 0$ and $y_1^{\cdot}(0) = 1$. On the above assumptions $a(t) \geqslant 0$, $f(x(t)) \leqslant 0$, and $P(t) \geqslant 0$ on $[0, {}^1/_2\theta]$, and $x^{\cdot}(0) = 0$ and $P(0) = 0$; hence $y_1^{\cdot}(t)$ cannot vanish at some point $t \leqslant {}^1/_2\theta$, since then the right-hand side of equality (2.3) is positive. Consequently, $y_1(t) > 0$ and $y_1^{\cdot}(t) > 0$ on $(0, {}^1/_2\theta]$. From this, using the identity $a(t) = a(\theta - t)$, we find that $y_1(t) > 0$ on $(0, \theta]$.

Let us consider the boundary value problem

$$y^{\cdot\cdot} + \lambda a(t) y = 0, \quad y(t_0) = y(\theta + t_0) = 0 \tag{2.4}$$

Since function $a(t)$ is θ-periodic, the Lebesque measure $L(u)$ of the set of values of t at which $a(t) \geqslant u$ is independent of t_0 on $[t_0, \theta + t_0]$, and since $a(t)$ increases on $[0, {}^1/_2\theta]$, and $a(t) = a(\theta - t)$, the minimum of the first eigen-value $\lambda_{\min}$ of problem (2.4) is reached in conformity with the theorem proved in /7/ at $t_0 = 0$. It was shown above that the solution $y_1(t)$ of Eq. (2.4) is positive on $(0, \theta]$ for $\lambda = 1$, hence $\lambda_{\min} > 1$. Consequently, the spacing of adjacent zeros of any solution of Eq. (1.5) exceeds θ.

In conformity with Adamov's theorem /8,9/ the necessary and sufficient condition of stability of the trivial solution of (1.5) is the fulfillment of one of the inequalities

$$D_n < \theta < d_{n+1} \tag{2.5}$$

where D_n and d_n are, respectively, the upper and lower bounds of spacing of some zero of any solution of Eq. (1.5) and the n-th next following zero. As shown above in this case $d_1 > \theta$ and $d_1 < \infty$ by virtue of the positiveness of $a(t)$. Hence the trivial solution of Eq. (1.5) is stable and, consequently, the solution $x(t, \mu)$ is asymptotically stable when condition (1.2) is satisfied and μ is small.

Let us prove that Eq. (1.4) has a unique solution of form (1.9). Let us assume that two such solutions, viz. $x_1(t)$ and $x_2(t)$ exist for some $\omega \in (\omega_*, \infty)$. Using the theorem on finite increments we find that the remainder $\Delta(t) = x_2(t) - x_1(t)$ satisfies the equation

$$\Delta^{\cdot\cdot} + \alpha(t)\Delta = 0, \quad \alpha(t) = f_x(\chi(t)), \quad \chi \in (x_1, x_2) \tag{2.6}$$

As previously shown, the spacing of two adjacent zeros of any solution $y(t)$ of Eq. (1.5) exceeds θ when $a(t) = a_1(t) = f_x(x_1(t))$ and $a(t) = a_2(t) = f_x(x_2(t))$. Since $f_x(x)$ is a nonincreasing function with respect to $|x|$, hence $\alpha(t) \leqslant a_1(t)$ for $|x_2(t)| > |x_1(t)|$ and $\alpha(t) \leqslant a_2(t)$ for $|x_2(t)| < |x_1(t)|$, and the spacing $\Delta(t)$ of zeros must also exceed θ. But by virtue of (1.9) $\Delta({}^1/_2\theta) = \Delta({}^3/_2\theta) = 0$. This contradiction proves the uniqueness of the solution of form (1.9)

Let us prove that $A(\omega)$ monotonically decreases. Setting in (1.4) $\omega t = \tau$ we obtain

$$\omega^2 x^{\cdot\cdot} + f(x) = p(\tau) \tag{2.7}$$

Let $x(A, \omega, \tau)$ be a solution of (2.7) for $x(A, \omega, 0) = -A$, $x^{\cdot}(A, \omega, 0) = 0$. The condition $x(A, \omega, {}^1/_2\pi) = 0$ which ensures the perodicity of solution also determines function $A(\omega)$ in implicit form.

Owing to the differentiability of $f(x)$, solution $x(A, \omega, \tau)$ is differentiable with respect to parameters A and ω, and $x_A = \partial x/\partial A$ and $x_\omega = \partial x/\partial\omega$ satisfy the equations

$$\omega^2 x_A^{\cdot\cdot} + a(\tau) x_A = 0, \quad x_A(0) = -1, \quad x_A^{\cdot}(0) = 0, \quad a(\tau) = f_x(x(A, \omega, \tau)) \tag{2.8}$$

$$\omega^2 x_\omega^{\cdot\cdot} + a(\tau) x_\omega = -2\omega x^{\cdot\cdot}(\tau), \quad x_\omega(0) = x_\omega^{\cdot}(0) = 0$$

When $x_A({}^1/_2\pi) \neq 0$ we have the derivative

$$\frac{dA}{d\omega} = -\frac{x_\omega({}^1/_2\pi)}{x_A({}^1/_2\pi)} \tag{2.9}$$

The solution of the second of Eqs. (2.8) may be written as

$$x_\omega(\tau) = -2\omega \int_0^\tau x^{\cdot\cdot}(s) y(\tau, s)\, ds = 2\omega \left[\int_0^\tau x^{\cdot}(s) y_s(\tau, s)\, ds - x^{\cdot}(s) y(\tau, s)\Big|_0^\tau\right], \quad y_s = \frac{\partial y(\tau, s)}{\partial s} \tag{2.10}$$

where $y(\tau, s)$ satisfies the first of Eqs. (2.8), $y(s, s) = 0$, and $y^{\cdot}(s, s) = 1$; $y_s(\tau, s)$ is also a solution of (2.8), and $y_s(s, s) = -1$ and $y_s^{\cdot}(s, s) = 0$, i.e. $y_s|(\tau, 0) = x_A(\tau)$.

The above analysis shows that $x_A(\tau) < 0$ on $[0, {}^1/_2\pi]$, hence $y_s(\tau, s) < 0$ when $0 \leqslant s \leqslant \tau \leqslant {}^1/_2\pi$ and $x_A({}^1/_2\pi) < 0$. Taking into account that $x^{\cdot}(\tau) > 0$ on $(0, {}^1/_2\pi]$, $x^{\cdot}(0) = 0$, we obtain from (2.10) and (2.9) that $x_\omega({}^1/_2\pi) < 0$ and $dA/d\omega < 0$. Thus $A(\omega)$ monotonically decreases on (ω_*, ∞).

To prove the inequality $A(\omega) > A_0(\omega)$ we consider the solution $x(A, \varepsilon, \tau)$, which depends on parameter ε, of the equation

$$\omega^2 x^{\cdot\cdot} + f(x) = \varepsilon p(\tau), \quad x(A, \varepsilon, 0) = -A, \quad x^{\cdot}(A, \varepsilon, 0) = 0$$

Function $A(\varepsilon, \omega)$ is determined by the condition $x(A, \varepsilon, {}^1/_2\pi) = 0$. Obviously $A_0(\omega) = A(0, \omega)$ and $A(1, \omega) = A(\omega)$. The derivative $x_\varepsilon = \partial x/\partial\varepsilon$ satisfies the equation

$$\omega^2 x_\varepsilon^{\cdot\cdot} + a(\tau)\, x_\varepsilon = p(\tau),\ x_\varepsilon(0) = x_\varepsilon^{\cdot}(0) = 0$$

Representing its solution in the form

$$x_\varepsilon(\tau) = \int_0^\tau p(s)\, y(\tau, s)\, ds = -\int_0^\tau P(s)\, y_s(\tau, s)\, ds + P(s)\, y(\tau, s)\,|_0^\tau$$

and taking into account (1.7) and the formulas obtained above, we obtain $x_\varepsilon(^1/_2\pi) > 0$. Consequently, $\partial A/\partial\varepsilon = -x_\varepsilon(^1/_2\pi)\cdot x_A^{-1}(^1/_2\pi) > 0$, i.e. $A(\varepsilon, \omega)$ monotonically increases as ε increases from 0 to 1. All statements of the theorem are thus proved.

Remark. 1°. If condition (1.8) is satisfied, then using (2.2) instead of (2.3) we can obtain the inequality $d_1 > \theta$ without assuming the monotonicity of $f(x)$. Therefore in this case the conclusions about the existence and uniqueness of solutions of form (1.9), as well as about those about the properties of $A(\omega)$ remain valid. The statement on stability of $x(t, \mu)$ is valid, if $d_1 < \infty$, i.e. when solutions of (1.5) are oscillatory.

Curve AB in Fig.1 represents the amplitude-frequency characteristic of the considered solution in conformity with the proved theorem.

When $\omega > \omega_0 = \sqrt{f_x(0)}$ ω_0 the frequency of "small" natural oscillations of the system, the following more general theorem applies.

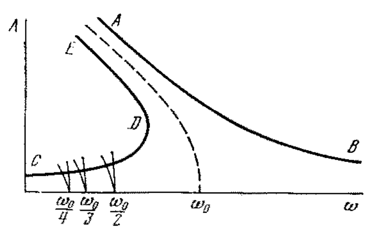

Fig.1

Theorem 2. System (1.4), (2.1) with mild nonlinearity has a solution of form (1.3), which is T-periodic and unique for any $\omega > \omega_0$. The respective solution of Eq. (1.1) with condition (1.2) and fairly small μ is asymptotically stable.

Proof. Since in a system with mild nonlinearity the natural oscillation period is $T_0(A) \geqslant 2\pi/\omega_0$, there exists in conformity with Opiala's theorem /4/ at least one T-periodic solution for $\omega > \omega_0$. If $x_2(t)$ is the second such solution, then from Eq. (2.6), where $0 \leqslant \alpha(t) \leqslant f_x(0) = \omega_0^2$, we obtain that $\Delta(t) = x_2(t) - x_1(t)$ oscillates and the spacing of zeros $d_1 \geqslant \pi/\omega_0 > \pi/\omega$. However the latter is impossible, since $\Delta(t)$ is periodic of period $2\pi/\omega$. This contradiction proves the uniquenss of $x_1(t)$.

Since $f(x) = -f(-x)$, $p(\omega t) = p(-\omega t)$, and $p(\omega t) = -p(\pi - \omega t)$, functions $-x_1(\pi/\omega - t)$ and $x_1(-t)$ satisfy Eq. (1.4). Owing to the uniqueness of solution $x_1(t)$ this means that $x_1(t) = -x_1(\pi/\omega - t)$ and $x_1(t) = x_1(-t)$, i.e. $x_1(t)$ is of form (1.3). Hence the respective coefficient of $a(t)$ in (1.5) is periodic of period $\theta = \pi/\omega < d_1$. This with allowance for (2.5) proves the last statement of the theorem.

Remark 2. Solution $x_1(t)$ is also of period nT, where n is any integer. Because of this, the existence and uniquenss of $x_1(t)$ implies that system (1.4), (2.1) with mild nonlinearity cannot have subharmonic oscillations of period $nT < 2\pi/\omega_0$.

3. Let us pass to the analysis of solutions of form (1.10).

Theorem 3. System (1.4) with mild nonlinearity has not more than two solutions of form (1.10) for any $\omega \in [\omega_0/3, \omega_0]$. When two such solutions exist, then with condition (1.8) or conditions (1.7) and (2.1) satisfied, the solution with the larger amplitude is unstable.

Proof. Let us assume that when $\omega > \omega_0/3$ there exist three solutions, viz. $x_1(t)$, $x_2(t)$, and $x_3(t)$ $(x_3(0) > x_2(0) > x_1(0))$ of form (1.10). Functions $\Delta_1(t) = x_2(t) - x_1(t)$ and $\Delta_2(t) = x_3(t) - x_2(t)$ satisfy Eq.(2.6) with coefficients $\alpha_1(t) = f_x(\chi_1(t)) \leqslant \omega_0^2$, $\alpha_2(t) = f_x(\chi_2(t)) \leqslant \omega_0^2$, where $\chi_1 \in (x_1, x_2)$, $\chi_2 \in (x_2, x_3)$. Taking into account that $\omega > \omega_0/3$ and $\Delta_1(t) = \Delta_1(-t)$, $\Delta_2(t) = \Delta_2(-t)$, and $\Delta_1(\pi/2\omega) = \Delta_2(\pi/2\omega) = 0$, we find that $\Delta_1 > 0$, $\Delta_2 > 0$ on $[0, \pi/(2\omega)]$, since otherwise the spacing of some adjacent zeros would be less than π/ω_0. Thus $x_3(t) > x_2(t) > x_1(t) \geqslant 0$ on $[0, \pi/(2\omega)]$ from which $\alpha_2(t) \leqslant \alpha_1(t)$, and $\alpha_2(t) \equiv \alpha_1(t)$ only when $f(x) = \omega_0^2 x$. This is, however, impossible, owing to the uniqueness of the solution. Hence the equalities $\Delta_1(\pi/(2\omega)) = 0$ and $\Delta_2(\pi/(2\omega)) = 0$ are incompatabile, and there are not more than two solutions of this type.

Let us prove the instability of $x_2(t)$. Since the respective coefficient $a_2(t) = f_x(x_2(t)) \leqslant \alpha_1(t)$ in (1.5), $y_2(t)$ $(y_2(0) = 1,\ y_2^{\cdot}(0) = 0)$ has no zeros on $[-\pi/(2\omega), \pi/(2\omega)]$. Consequently, the upper bound of spacing of adjacent zeros $D_1 > \theta$. Let us show that the lower bound $d_1 < \theta$.

We select on $[0, \pi/(2\omega))$ point t_1 such that $x_2^{\cdot}(t_1) = 0$, $x_2^{\cdot}(t) < 0$ for $t_1 < t \leqslant \pi/(2\omega)$ and consider the solution $y(t)$ $(y(t_1) = 0,\ y^{\cdot}(t_1) = 1)$ of Eq. (1.5). Since $x(\pi/(2\omega)) = 0$, $x^{\cdot}(t_1) = 0$, and $x^{\cdot}(\pi/(2\omega)) < 0$, hence, setting in (2.2) or (2.3) the lower bound at t_1 and the upper equal $-\pi/(2\omega)$, then, taking into account conditions (1.8) or (1.7) and (2.1), respectively, we find that $y^{\cdot}(t)$ has not less than one zero on $[t_1, \pi/(2\omega)]$, since otherwise equalities (2.2) or (2.3) would have their two sides of different sign. If $y^{\cdot}(\pi/(2\omega)) < 0$, then identity

$a(t) = a(\pi/w - t)$ would imply that $y(t)$ has a zero on $[\pi/(2\omega), \pi/\omega]$. If $y^{\cdot}(\pi/(2\omega)) > 0$, then $y^{\cdot}(t)$ has not less than two zeros on $[t_1, \pi/(2\omega)]$, with point t_2 between them, at which $y^{\cdot\cdot}(t_2) = 0$, and by virtue of (1.5) $y(t_2) = 0$ or $a(t_2) = 0$. In the latter case $a(t) \leqslant 0$ on $[t_1, t_2]$, because $x_2(t)$ decreases and $f_x(x_2(t))$ increases on $[t_1, t_2]$. Consequently, when $y(t) > 0$, then by virtue of (1.5) $y^{\cdot\cdot}(t) \geqslant 0$ on $[t_1, t_2]$, and $y^{\cdot}(t)$ cannot have zeros there. This contradiction shows that $y(t_2) = 0$

Hence $d_1 < \theta < D_2$ which according to criterion (2.5) shows the instability of solution $x_2(t)$. The theorem is proved.

If for $\omega < \omega_0/3$, the inequality $|x_2(t)| \geqslant |x_1(t)|$ holds for solutions of form (1.10), the statement about the instability of $x_2(t)$ remains valid, since all of the above reasoning applies in this case.

It can be shown that when $p(\omega t) > 0$ on $[0, \pi/(2\omega)]$, the amplitude-frequency characteristic $A(\omega)$ of a solution of the form (1.10) lies to the left of the skeleton curve. When $p = p_1 \cos \omega t$, $A(\omega)$ is of the form of curve CDE (Fig.1). Solutions that correspond to the DE branch are in conformity with Theorem 3 unstable.

Note that the stability of branch AB and instability of DE for a quasilinear system subjected to harmonic effects is well-known (see /1,3/ and other). Theorems 1— 3 extend these conclusions to a wide class of perturbing effects, without imposing constraints on their magnitude, and on the degree of a system nonlinearity.

Let $x(t)$ be a solution of form (1.3). Taking into account that $f_x(A) \leqslant a(t) \leqslant \omega_0^2$ and applying the Joukowski criterion /10/, we obtain the sufficient conditions for stability of the trivial solution of (1.5)

$$\frac{\omega_0}{n+1} \leqslant \omega \leqslant \frac{\sqrt{f_x(A)}}{n}, \quad n = 1, 2, 3, \ldots \tag{3.1}$$

If these inequalities are satisfied for some n, the solution $x(t, \mu)$ with condition (1.2) satisfied and small μ is asymptotically stable. Conditions (3.1) may be used for analyzing the stability of low-amplitude oscillations at low frequency (e.g., those corresponding to branch CD in Fig.1).

Since for $a(t) \neq \mathrm{const}$ $a_n \neq D_n$, the amplitude-frequency characteristic $A(\omega)$ always contains regions where the stability conditions (2.5) are not satisfied. As implied by (3.1), they lie to the left of ω_0/n and contract to them as $A \to 0$ (Fig.1). Note that quasilinear and quasi-Liapunov systems do not have such instability regions.

4. We pass to the investigation of systems with strict nonlinearity. First, we consider solutions of form (1.10). As shown above, they exist under condition (1.7) for any A, and $\lim \omega(A) = \infty$ as $A \to 0$ and $\lim \omega(A) = \lim \omega_0(A) = k$ as $A \to \infty$, where $k^2 = \lim [f(A) A^{-1}]$ as $A \to \infty$ /4/.

Let $y_1(t)$ $(y_1(0) = 0,\ y_1^{\cdot}(0) = 1)$, $y_2(t)$ $(y_2(0) = 1$ and $y_2^{\cdot}(0) = 0)$ be solutions of Eq. (1.5). As shown in the proof of Theorem 1, $y_1(t) > 0$ on $(0, \pi/\omega]$ independently of the character of nonlinearity, hence $y_2(t)$ has not more than one zero on $[0, \pi/(2\omega)]$. Consequently, if $x_A(\pi/2) = -y_2(\pi/(2\omega)) = 0$, then in (2.10) $y_s(\tau, s) < 0$ for $0 \leqslant s \leqslant \tau < \pi/2$, and as the result, $x_\omega(\pi/2) < 0$. The amplitude-frequency characteristic $A(\omega)$, defined by the equation $x(A, \omega, {}^1/_2\pi) = 0$, is therefore represented by a smooth curve free of singular points, and there exists for any $\omega \in (\omega_*, \infty)$ $(\omega_* = \inf \omega(A))$ at least one solution of form (1.10). If $k > \omega_*$ and $\omega_* < \omega < k$ the number of such solutions exceeds one.

Theorem 4. For $\omega_0 < \omega < k$ system (1.4), (1.7) with strict nonlinearity has two solutions: $x_1(t)$ and $x_2(t)$ $(A_1 < A_2)$ of form (1.9), and for $\omega > k$ it has the unique solution $x_1(t)$. Solution $x_2(t)$ is unstable, while solution $x(t, \mu)$ which corresponds to $x_1(t)$ is asymptotically stable if condition (1.2) is satisfied and μ fairly small.

Proof. Let $x_1(t)$ and $x_2(t)$ be solutions of form (1.9) for $\omega \in (\omega_*, k)$. Since $f_x(x)$ increases (does not decrease) with $|x|$, hence for $|x_2| > |x_1|$

$$a_1(t) \leqslant \alpha_1(t) \leqslant a_2(t), \quad a_1(t) \quad f_x(x_1(t)), \quad a_2(t) = f_x(x_2(t)) \tag{4.1}$$

As previously shown, $y_1(t) > 0$ on $(0, \theta]$ for $a(t) = a_1(t)$ and $a(t) = a_2(t)$, hence in both cases $D_1 > \theta$. This also shows that $y_2(t)$, as well as $\Delta(t)$ (by virtue of the first of inequalities (4.1)) have on $[0, \theta]$ not more than one zero. Since $\Delta(-\theta/2) = \Delta(\theta/2) = 0$, $y_2(t)$ has two zeros on $(-\theta/2, \theta/2)$ when $a(t) = a_2(t)$ and, consequently, the inequality $d_1 < \theta < D_1$ holds for $a(t) = a_2(t)$, i.e. solution $x_2(t)$ is unstable.

The left-hand inequality in (4.1) implies that $y_2(t)$ has no zeros on $[-\theta/2, \theta/2]$ when $a(t) = a_1(t)$. Since $a_1(t)$ monotonically decreases (does not increase) on $[0, \theta/2]$, and $a_1(t) = a_1(-t)$, hence, as shown in the proof of Theorem 1, the spacing of adjacent of any solution also exceeds θ. Thus $d_1 > 0$ when $a(t) = a_1(t)$ and, consequently, solution $x_1(t, \mu)$ is asymptotically stable when condition (1.2) is satisfied and μ is fairly small.

Since for any one of the two solutions $x_1(t)$ and $x_2(t)$ we have $d_1 > \theta$ and for the other

$-d_1 < \theta$, the number of solutions for $\omega \in (\omega_*, k)$ is equal two.

To prove the theorem when $k < \infty$ and $\omega > k$ we consider a system with rigid characteristic $f^*(x)$, such that $f^*(x) = f(x)$ for $0 \leqslant x \leqslant x^*$, $k^{*2} = \lim f^*(A) A^{-1} = \infty$ as $A \to \infty$. For the corresponding solution $x_1^*(t)$ the theorem holds for any ω. But $x_1(t) = x_1^*(t)$ when $A < x^*$, and x^* may be chosen arbitrarily large, therefore the theorem is also valid for solution $x_1(t)$. The above reasoning evidently holds also in the case of $k = \omega_*$, i.e. when solution $x_2(t)$ does not exist. The theorem is proved.

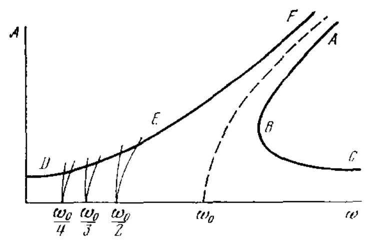

Fig.2

Thus the amplitude-frequency characteristic $A(\omega)$ has two equivalent branches $A_1(\omega)$ and $A_2(\omega)$ that correspond to solutions $x_1(t)$ and $x_2(t)$. As shown in the proof of Theorem 1, $x_1(t)$ monotonically decreases. If $p(\omega t) \geqslant 0$ on $[0, \pi/(2\omega)]$, it is possible to prove similarly that $A_2(\omega)$ monotonically increases and satisfies the inequality $A_2(\omega) < A_0(\omega)$, i.e. $A(\omega)$ has the form of curve ABC (Fig.2). For a less strict condition (1.7) branch AB may, generally, intersect the skeleton curve, i.e. the amplitudes of unstable solutions may exceed those of natural oscillations.

The Joukowski criterion /10/ can be used in the analysis of stability of low frequency and amplitude periodic oscillations which for $\mu = 0$ are of form (1.3). Taking into account that $f_x(A) \geqslant a(t) \geqslant \omega_0^2$, we obtain

$$\frac{\sqrt{f_x(A)}}{n+1} \leqslant \omega \leqslant \frac{\omega_0}{n}, \quad n = 1, 2, 3, \ldots \tag{4.2}$$

The fulfillment of these inequalities for some n ensures the stability of the trivial solution of (1.5) and, when condition (1.2) is satisfied and μ fairly small, also, that of solution $x(t, \mu)$. It follows from (4.2) that the instability regions lie to the right of ω_0/n to which they contract as $A \to 0$ (Fig.2). Note that such instability regions in a system defined by the Duffing equation were investigated in /6/.

Let us consider solution of form (1.10). It was shown in Sect.1 that for $A \geqslant A_*$ function $x(t)$ and, consequently, the respective coefficient $a(t)$ monotonically decrease on $[0, \pi/(2\omega)]$. Hence from the theorem in /7/ follows that the maximum of the first eigenvalue λ_{max} of the boundary value problem is reached at $t_0 = 0$. It was established in the proof of Theorem 3 that when condition (1.7) is satisfied, solution $y_1(t)$ $(y_1(0) = 0, y_1^{\cdot}(0) = 1)$ of Eq. (1.5) has a zero on $(0, \theta)$. Hence $\lambda_{max} < 1$ and, consequently, the upper bound of the spacing of adjacent zeros $D_1 < \theta$.

Equation (2.2) implies that when $p(\omega t) \equiv 0$, then $y_1^{\cdot}(\theta/2) = 0$ from which $y_1(\theta) = 0$, i.e. for natural oscillations we have $D_1^{\circ} = \theta$. As ω is increased $A(\omega) \to A_0(\omega)$ and $D_1 \to D_1^0$, which shows that from some ω the stability condition $D_1 < \theta < d_2$ becomes satisfied. When conditions (1.2) and (1.7) are satisfied and μ is fairly small, solution $x(t, \mu)$, which corresponds to resonance oscillations of form (1.10) (curve EF in Fig.2), is thus asymptotically stable. Since $a(t) \leqslant f_x(A)$, the sufficient condition of stability (which ensures the fulfillment of inequality $\theta < d_2$) is of form $\omega \geqslant {}^1/_2 \sqrt{f_x(A)}$.

When $p(\omega t)$ increases with ω fixed, $x(t)$ and the coefficient $a(t)$ in (1.5) increase, while d_n and D_n decrease and, consequently, the conditions of stability (if it exists) and instability of solution (1.10) will be successively satisfied. Note that in the case of Duffing's equation the alternation of stable and unstable solutions, as perturbing force increases, was theoretically and experimentally studied in /5/.

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